Real Analysis Qual Prep Summer 2009

Assignment 2: Functions of BV and Measures

1. Assume $\{f_n\}$ is a sequence of real-valued nondecreasing functions on $I = [a, b]$, and suppose $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in I$. Is necessarily nondecreasing?

2. Show that a monotone function $f : [a, b] \to \mathbf{R}$ has at most countably many discontinuities, and they are all of the first kind. Conversely if D is an at most countable subset of $[a, b]$, construct a monotone function whose set of discontinuities is precisely D.

3. A real-valued function f defined on $I = [a, b]$ is said to be Lipschitz if there exists a constant c such that $|f(x) - f(y)| \leq c|x - y|$ for all $x, y \in I$. Show that if f is Lipschitz on I, it is BV there as well.

4. Let $f, g \in BV(I)$, with $I = [a, b]$. Show that $fg \in BV(I)$, and that if $|g(x)| \geq \epsilon > 0$ for all $x \in I$, then $f/g \in BV(I)$. Estimate $V(fg; a, b)$ and $V(f/g; a, b)$ in terms of $V(f; a, b)$, $V(g; a, b)$, and ϵ.

5. Let $\{q_1, q_2, ...\}$ be an enumeration of $\mathbf{Q} \cap (0, 1)$. Define $f : [0, 1] \to \mathbf{R}$ by $f(x) = \begin{cases} 2^{-n} & , x = q_n \\ 0 & , \text{otherwise} \end{cases}$
Show that f has bounded variation.

6. Is $f(x) = x\sin(\frac{1}{x})$ absolutely continuous? Is it of bounded variation? What about $g(x) = x^2 \sin(\frac{1}{x})$?

7. Let $(X, \mathcal{S}, \mu)$ be a measure space. For $A, B \subset X$, let

$$A\Delta B = (A\backslash B) \cup (B\backslash A).$$

   (a) If μ is complete, $A \in \mathcal{S}$, $B \in X$, $\mu(A\Delta B) = 0$, show that $B \in \mathcal{S}$ and $\mu(B) = \mu(A)$.

   (b) If A, $B \in \mathcal{S}$ and $\mu(A\Delta B) = 0$, show $\mu(A) = \mu(B)$.

   (c) For $A, B \in \mathcal{S}$, let $A \sim B$ if $\mu(A\Delta B) = 0$. Prove that $\sim$ is an equivalence relation on $\mathcal{S}$.

   (d) If $\mu(X) < \infty$, let $[A] \in \mathcal{S}/\sim$ and $[B] \in \mathcal{S}/\sim$ be two classes of equivalence on $\mathcal{S}/\sim$. Define

$$d([A], [B]) = \mu(A\Delta B),\ A \in [A],\ B \in [B].$$

   Prove that d is a metric on $\mathcal{S}/\sim$.

8. Let $E \subset \mathbf{R}$ be a Lebesgue measurable set and let $\delta > 0$. If μ is the Lebesgue measure and $\mu(E \cap (a,b)) \geq \delta(b-a)$ for all intervals $(a,b) \subset \mathbf{R}$, show that $\mu(\mathbf{R} \backslash E) = 0$.

9. Prove that

$$\limsup_n (A_n \cap B_n) \subseteq (\limsup_n A_n) \cap (\limsup_n B_n),$$

and that

$$(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n).$$

What are the corresponding statements for the lim inf?

10. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $E_n \subseteq \mathcal{F}$. Show that if

$$\mu(\cup_n E_n) < \infty, \text{ and } \mu(E_n) \geq \eta > 0 \text{ for infinitely many } \eta\text{'s},$$

then $\mu(\limsup_n E_n) > 0$. Find an example to show that the condition $\mu(\cup_n E_n) < \infty$ cannot be removed.

11. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $E_n \subseteq \mathcal{F}$. Show that

$$\mu(\liminf_n E_n) \leq \liminf_n \mu(E_n),$$

and, provided $\mu(\cup_n E_n) < \infty$,

$$\limsup_n \mu(E_n) \leq \mu(\limsup E_n).$$

By means of example show that we may have strict inequalities above.

12. Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Fix $1 \leq n \leq m$, and let $E_1, \ldots, E_m$ be measurable sets with the property that for almost every $x \in X$ belongs to at least n of these sets. Prove that at least one of these sets must have μ-measure greater than or equal to n/m.

13. Suppose A, B are not Lebesgue measurable. Is the same true of $A \cup B$?

14. Suppose that E is a Lebesgue measurable subset of $\mathbf{R}$ such that $m(E) < \infty$. Define $f(x) = m((E + x) \cap E)$. Prove that f is a continuous function on $\mathbf{R}$ and that $\lim_{x \to \infty} f(x) = 0$.

15. Let $A \subseteq [0,1]$ be a measurable set of positive measure. Show that there exist $x \neq y \in A$ such that $x - y \in \mathbf{Q}$.

16. Let A and B be (not necessarily Lebesgue measurable) subsets of $\mathbf{R}$ and let $|\cdot|_e$ stand for Lebesgue outer measure.
Prove that if $|A|_e = 1$ and $|B|_e = 1$ and $|A \cup B|_e = 2$ then $|A \cap B|_e = 0$.

17. If $-1 \leq r \leq 1$, show there exist elements of the Cantor set x, y such that $x - y = r$.

18. Let $A \subset \mathbf{R}$ be a Lebesgue measurable set. Show that if $0 \leq b \leq m(A)$, then there is a Lebesgue measurable set $B \subset A$ with $m(B) = b$.

19. Prove that there exists a Lebesgue measurable set $E \subset \mathbf{R}$ such that

$$0 < |E \cap I| < |I|, \text{ for all bounded intervals } I \subset \mathbf{R}.$$