ECE 301: Homework 7

SOLUTIONS

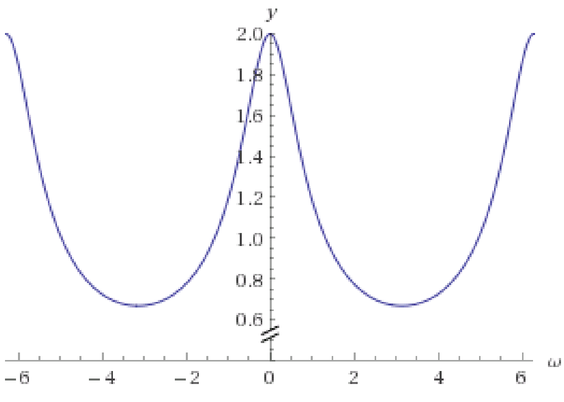
Problem 1

$X(\omega) = \sum_n x[n]e^{-j\omega n}$

Part a:

$x[n] = (\frac{1}{2}^{n-1}u[n-1]$ $X(\omega) = \sum_{n=1}^{\infty} \frac{1}{2}^{n-1} e^{-j\omega n}$
Let l = n-1
n = l+1
$= \sum_{l=0}^{\infty} \frac{1}{2}^{n-1} e^{-j\omega(l+1)} = e^{-j\omega} \sum_{l=0}^{\infty} (\frac{1}{2}e^{-j\omega})^l = e^{-j\omega}\frac{1}{1-\frac{1}{2}e^{-j\omega}}$

$$= \frac{e^{-j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$



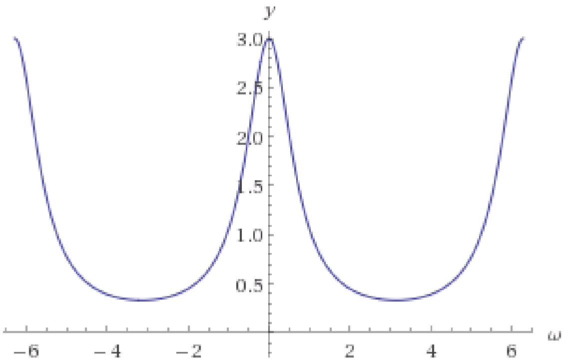
(ω from −6.3 to 6.3)

Part b:

$x[n] = (\frac{1}{2})^{|n-1|}$
$X(\omega) = \sum_{n=-\infty}^{\infty} (\frac{1}{2})^{|n-1|} e^{-j\omega} = \sum_{m=-\infty}^{\infty} (\frac{1}{2})^{|m|} e^{-j\omega(m+1)}$
Let m = n-1
n=m+1
$= e^{-j\omega}[\sum_{m=0}^{\infty} (\frac{1}{2})^m e^{-j\omega m} + \sum_{m=-\infty}^{-1} (\frac{1}{2})^{-m} e^{-j\omega m}]$
Let l=-m
$= e^{-j\omega}[\sum_{m=0}^{\infty} (\frac{1}{2}e^{-j\omega})^m + \sum_{l=1}^{\infty} (\frac{1}{2}e^{j\omega})^l]$

$= e^{-j\omega}[\frac{1}{1-\frac{1}{2}e^{-j\omega}} + \sum_{l=0}^{\infty}(\frac{1}{2}e^{j\omega})^l - 1]$
$=...=$

$$= \frac{3e^{-j\omega}}{5 - 4\cos(\omega)}$$



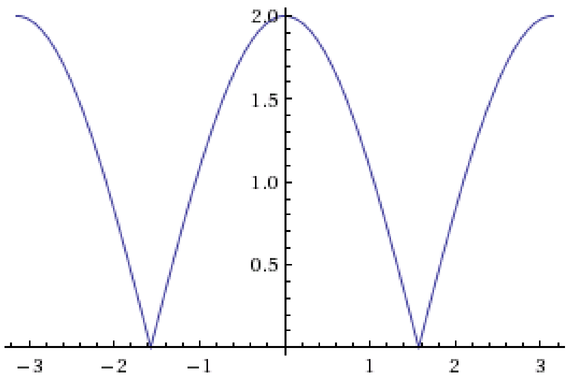
(ω from −6.3 to 6.3)

Problem 2

Part a:

$x[n] = \delta[n-1] + \delta[n+1]$
$X(\omega) = \sum_n(\delta[n-1] + \delta[n+1])e^{-j\omega n} = e^{-j\omega n}|_{n=1} + e^{-j\omega n}|_{n=-1}$ from the two δ's
$= e^{-j\omega} + e^{j\omega}$

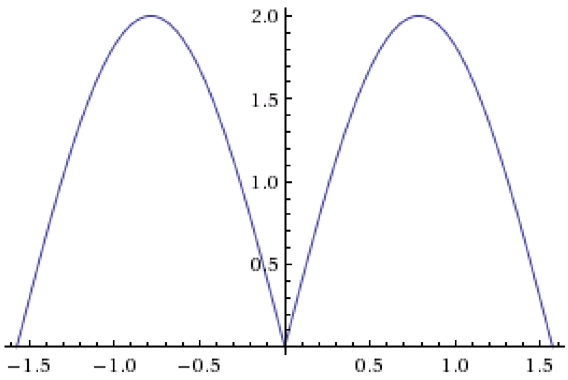
$$= 2\cos(\omega)$$



Part b:

$x[n] = \delta[n+2] - \delta[n-2]$
$X(\omega) = \sum_n(\delta[n+2] - \delta[n-2])e^{-j\omega n} = e^{-j\omega n}|_{n=-2} - e^{-j\omega n}|_{n=2}$ from the two δ's
$= e^{j2\omega} - e^{-2j\omega}$

$$= 2j\sin(2\omega)$$



Problem 3

Part c:

$$x[n] = (\frac{1}{3})^{|n|}u[-n-2]$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{-2} (\frac{1}{3})^{|n|}e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-2} (\frac{1}{3})^{-n}e^{-j\omega n}$$

$$= \sum_{k=0}^{\infty} (\frac{1}{3}e^{j\omega})^{-k-2}$$

Let $k = -n - 2$

$$= (\frac{1}{3}e^{j\omega})^{-2}\frac{1}{(1-\frac{1}{3}e^{j\omega})}$$

Part h:

$$x[n] = sin(\frac{5\pi}{3}n) + cos(\frac{7\pi}{3}n) = sin(\frac{5\pi}{3}n - 2n\pi) + cos(\frac{7\pi}{3}n - 2n\pi)$$

$$= sin(\frac{-\pi}{3}n) + cos(\frac{\pi}{3}n) = -sin(\frac{\pi}{3}) + cos(\frac{\pi}{3}n)$$

$$= \pi \sum_{l=-\infty}^{\infty} (\delta(\omega - \frac{\pi}{3} - 2l)(1 - \frac{1}{j}) + \delta(\omega + \frac{\pi}{3} - 2l)(1 + \frac{1}{j}))$$

$$= \pi \sum_{l=-\infty}^{\infty} ((1+j)\delta(\omega - \frac{\pi}{3} - 2\pi l) + (1-j)\delta(\omega + \frac{\pi}{3} - 2\pi l))$$

Part j:

$$x[n] = (n-1)(\frac{1}{3})^{|n|}$$

Let $y[n] = (\frac{1}{3})^{|n|}$

$$Y(e^{jw}) = \sum_{n=-\infty}^{\infty}((\frac{1}{3})^{|n|}e^{-j\omega n}) = \sum_{n=-\infty}^{0}((\frac{1}{3})^{-n}e^{-j\omega n}) + \sum_{n=0}^{\infty}((\frac{1}{3})^{n}e^{-j\omega n}) - (\frac{1}{3})^{n}e^{-j\omega n}|_{n=0}$$

$$= \sum_{n=0}^{\infty}(\frac{1}{3}e^{j\omega})^n + \sum_{n=0}^{\infty}(\frac{1}{3}e^{-j\omega})^n - 1$$

$$= \frac{1}{1-\frac{1}{3}e^{j\omega}} + \frac{1}{1-\frac{1}{3}e^{-j\omega}} - 1$$

$$= \frac{1-\frac{1}{3}e^{-j\omega}+1-\frac{1}{3}e^{j\omega}}{1-\frac{1}{3}e^{-j\omega}-\frac{1}{3}e^{j\omega}+\frac{1}{9}} - 1$$

$$= \frac{2-\frac{2}{3}cos(\omega)}{\frac{10}{9}-\frac{2}{3}cos(\omega)} - 1$$

$$= \frac{2-\frac{2}{3}cos(\omega)-\frac{10}{9}+\frac{2}{3}cos(\omega)}{\frac{10}{9}-\frac{2}{3}cos(\omega)}$$

$$= \frac{\frac{8}{9}}{\frac{10}{9}-\frac{2}{3}cos(\omega)} = \frac{8}{10-6cos(\omega)} = \frac{4}{5-3cos(\omega)}$$

$$x[n] = ny[n] - y[n]$$
$$X(e^{j\omega}) = j\tfrac{d}{d\omega}Y(e^{j\omega}) - Y(e^{j\omega})$$

$$Y(e^{j\omega}) = 4(5-3cos(\omega))^{-1}$$

$$= j[-4(5-3cos(\omega))^{-2}(3sin(\omega))] - \frac{4}{5-3cos(\omega)}$$

$$= \frac{-20-12(cos(\omega)+jsin(\omega))}{(5-3cos(\omega))^2}$$

$$= \frac{-20-12e^{j\omega}}{(5-3cos(\omega))^2}$$

Problem 4

Part a:

$$X(e^{j\omega}) = \begin{cases} 1 & \text{if } \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0 & \text{if } ow \end{cases}$$

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\pi}^{-\pi} X(e^{j\omega})e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}(\frac{1}{jn}e^{j\omega n}|_{\frac{-3\pi}{4}}^{\frac{-\pi}{4}} + \frac{1}{jn}e^{j\omega n}|_{\frac{\pi}{4}}^{\frac{3pi}{4}}) \\
&= \frac{1}{2\pi}\frac{1}{jn}(e^{-j\frac{\pi}{4}n} - e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n} - e^{tj\frac{\pi}{4}n}) \\
&= \frac{1}{\pi n}(\sin(\frac{3\pi}{4}n) - \sin(\frac{\pi}{4}n))
\end{aligned}
$$

$$
= \frac{\sin(\frac{3\pi}{4}n)}{\pi n} - \frac{\sin(\frac{\pi}{4}n)}{\pi n}
$$

Part b:

$$
X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} - 4e^{-j\omega} + e^{-j10\omega}
$$

$$
x[n] = \delta(n) + 3\delta(n-1) + 2\delta(n-2) - 4\delta(n-3) + \delta(n-10)
$$

Part c:

$$
\begin{aligned}
X(e^{j\omega}) &= e^{-j\frac{\omega}{2}} \\
x[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-j\frac{\omega}{2}}e^{tj\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{pi} e^{tj(n-\frac{1}{2})\omega}d\omega \\
&= \frac{1}{j(n-\frac{1}{2})}e^{j(n-\frac{1}{2})\omega)}|_{-\pi}^{\pi} \\
&= \frac{1}{j(n-\frac{1}{2})}((-1^n)(-j) - (-1)^n(j))
\end{aligned}
$$

$$
= \frac{-2(-1)^n}{n-\frac{1}{2}}
$$

Problem 5

Part a:

$$
X(e^{jo}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j(o)n} = \sum_{n=-\infty}^{\infty} x[n]) = -1 + 1 + 2 + 1 + 1 + 2 + 1 - 1
$$

$$
= 6
$$

Part c:

$$\int_{-\pi}^{pi} X(e^{j\omega})d\omega = \int_{-\pi}^{\pi} X(e^{j\omega})e^{j(0)\omega}d\omega = 2\pi x[0]$$

$$\boxed{= 4\pi}$$

Part d:

$$X(e^{j\pi}) = \textstyle\sum_{n=-\infty}^{\infty} x[n]e^{-j(\pi)n} = \sum_{n=-\infty}^{\infty}(-1)^n x[n]$$
$$= (-1)(-1)+(1)(0)+(-1)(1)+(1)(2)+(-1)(1)+(1)(0)+(-1)(1)+(1)(2)+(-1)(1)+(1)(0)+(-1)(-1)$$

$$\boxed{= 2}$$

Part f:

(i)

$$\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{-\infty}^{\infty} \left|X(e^{j\omega})\right|^2$$
$$= 2\pi((-1)^2 + 1^2 + 2^2 + 1^2 + 1^2 + 2^2 + 1^2 + (-1)^2)$$

$$\boxed{= 28\pi}$$

(ii)

$$\int_{-\pi}^{\pi} \left|\frac{d}{d\omega}X(e^{j\omega})\right|^2 d\omega = \int_{-\pi}^{\pi} \left|j\frac{d}{d\omega}X(e^{j\omega})\right|^2 d\omega = 2\pi \sum_{n=-\infty}^{\infty} \left|X(e^{j\omega})\right|^2$$
$$= 2\pi \sum_{n=-\infty}^{\infty} |nx[n]|^2$$
$$= 2\pi(3^2 + (-1)^2 + 1^2 + 3^2 + 8^2 + 5^2 + (-7)^2)$$

$$\boxed{= 316\pi}$$

Problem 6

$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 2\pi X[0]$$
$$\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0 <=> X[0] = 0$$
$$X(e^{j(\omega+2\pi)} = \sum_{n=-\infty}^{\infty} X[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} X[n]e^{-j\omega n}$$

$$=> X(e^{j\omega}) \text{ is always periodic with period } 2\pi$$
$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$
$$X(e^{j0}) = 0 <=> \sum_{n=-\infty}^{\infty} x[n] = 0$$

Part b:

$$x[n] = 0 => \int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0$$
$$X(e^{j\omega}) \text{ is periodic}$$
$$\sum_{n=-\infty}^{\infty} x[n] = 0 => X(e^{j0}) = 0$$

Part c:

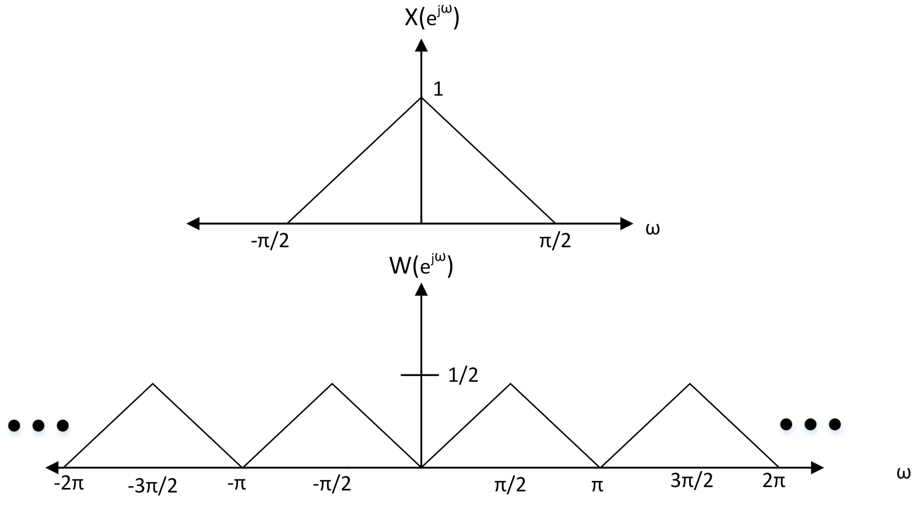
$$x[n] = (\tfrac{1}{2})^n u[n]$$
$$x[0] = 1 => \int_{-\pi}^{\pi} X(e^{j\omega})d\omega \neq 0$$
$$X(e^{j\omega}) \text{ is periodic}$$
$$\sum_{n=-\infty}^{\infty} x[n] = \frac{1}{1-\frac{1}{2}} \neq 0 => X(e^{j\omega}) \neq 0$$

Part e:

$$x[n] = \delta[n-1] + \delta[n+2]$$
$$x[0] = 0 => \int_{-\infty}^{\infty} X(e^{j\omega})d\omega = 0$$
$$X(e^{j\omega}) \text{ is periodic}$$
$$\sum_{n=-\infty}^{\infty} x[n] = 2 \neq 0 => X(e^{j\omega}) \neq 0$$

Problem 7

Part a-ii:

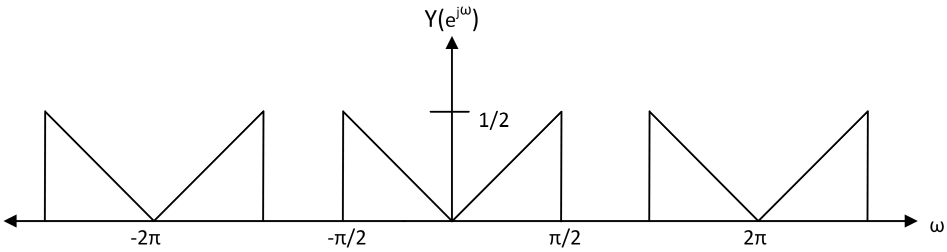


$$p[n] = \cos(\tfrac{\pi n}{2})$$
$$w[n] = x[n]p[n]$$

$$W(e^{j\omega}) = \frac{1}{2}X(e^{j(\omega-\frac{\pi}{2})}) + \frac{1}{2}X(e^{j(\omega+\frac{\pi}{2})})$$

Part b-ii:

$h[n] = \frac{\sin(\frac{\pi n}{2})}{\pi n}$
$H(e^{j\omega})$=1 when $|\omega| \leq \frac{\pi}{2}$ and 0 otherwise. Note: This is repeated every 2π.



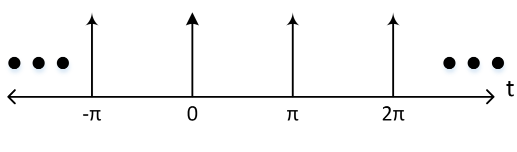
$$y[n] = \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{0} \frac{-1}{\pi}\omega e^{-j\omega n}dw + \frac{1}{2\pi}\int_{\frac{-\pi}{2}}^{0} \frac{1}{\pi}\omega e^{-j\omega n}dw$$
$$u = w$$
$$du = dw$$
$$dv = e^{j\omega n}dw$$
$$v = \frac{1}{jn}e^{j\omega n}$$

$$= \frac{1}{2\pi}[-\frac{1}{\pi}(\frac{\omega}{jn}e^{-j\omega n}|_{-\frac{\pi}{2}}^{0} - \frac{1}{jn}\int_{\frac{-\pi}{2}}^{0} e^{j\omega n}dw) + \frac{1}{\pi}(\frac{\omega}{jn}e^{j\omega n}|_{0}^{\frac{\pi}{2}} - \frac{1}{jn}int_0^{\frac{\pi}{2}} e^{j\omega n}dw)]$$
$$= \frac{1}{2\pi}[-\frac{1}{\pi}(\frac{\pi}{j2n}e^{-j\frac{\pi}{2}n} - \frac{1}{(jn)^2}e^{j\omega n}|_{-\frac{\pi}{2}}^{0}) + \frac{1}{\pi}(\frac{\pi}{j2n}e^{-j\frac{\pi}{2}n} - \frac{1}{(jn)^2}e^{j\omega n}|_{0}^{-\frac{\pi}{2}})]$$
$$= \frac{1}{2\pi}[\frac{1}{j2n}(e^{j\frac{\pi}{2}n} - e^{-j\frac{\pi}{2}n}) + \frac{1}{\pi n^2}(e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}) - \frac{1}{\pi n^2} - \frac{1}{\pi n^2}]$$
$$= \frac{1}{2\pi}[\frac{\sin(\frac{\pi}{2}n)}{n} + \frac{2\cos(\frac{\pi}{2}n)}{\pi n^2} - \frac{2}{\pi n^2}]$$

$$= \frac{\sin(\frac{\pi}{2}n)}{2\pi n} + \frac{\cos(\frac{\pi}{2}n)}{\pi^2 n^2} - \frac{1}{\pi^2 n^2}$$

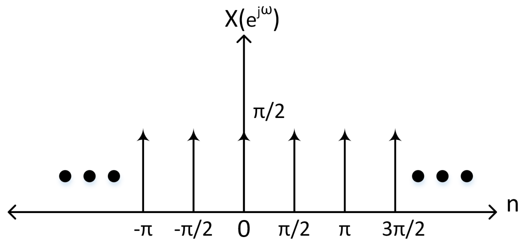
Problem 8

$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-4k]$



x[n] is periodic (N=4)

Find the DTFS coefficients, a_k
x[n]=$\delta[n]$ for $0 \leq n \leq 3$
$a_k = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}kn}$
$= 1/4$



$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi}{N}k)$

$$= \sum_{k=-\infty}^{\infty} \frac{\pi}{2}\delta(\omega - \frac{\pi}{2}k)$$